COMSATS Institute of Information Technology,
Islamabad Campus

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Experiment 9: Compound Pendulum

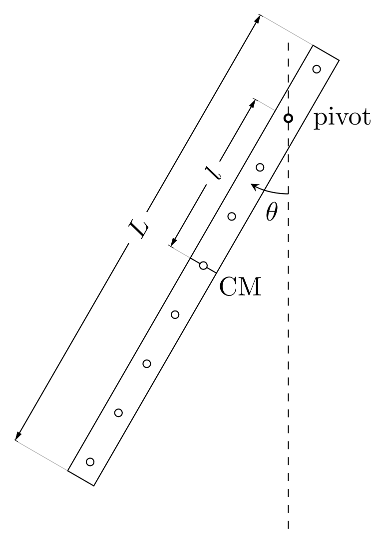
A compound pendulum (also known as a physical pendulum) consists of a **rigid** body oscillating about a pivot. This experiment uses a uniform metallic bar with holes/slots cut down the middle at regular intervals. The bar can be hung from any one of these holes allowing us to change the location of the pivot.

Objective

Derive an equation for the time period T of the oscillations of a uniform metallic bar suspended from a pivot passing through it.

Experimental Setup

The experimental equipment consists of a thin uniform metallic bar with holes/slots placed through it at regular intervals. By allowing the bar to swing from different slots one can change the moment of inertia and consequently the Time Period of oscillations.



We define the total length of the bar as L and the distance from the pivot to the center of mass (CM) of the bar to be l as indicated in the diagram above. The position of the bar at any instant of time is given by the angle θ. When allowed to swing the bar performs an approximation of simple harmonic motion, that is, the angle θ varies in a cyclic fashion with time period T.

Free-Body Diagram

To calculate the time period T one has to derive the equation of motion $\theta(t)$, namely how the angle θ varies as a function of time t. The first step, as always, is drawing the extended free body diagram of the system (extended because we are dealing with a rotational system and therefore the distance from the pivot is significant).

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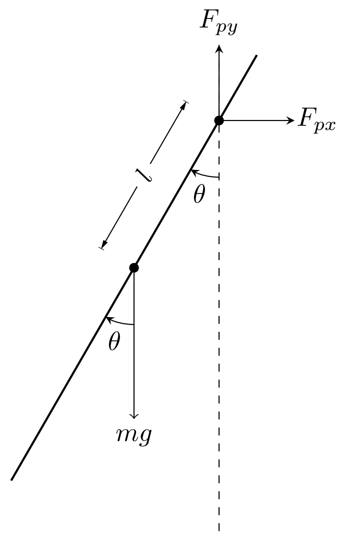


There are two forces acting on the bar. Its weight, which acts at the bar's center of mass/gravity, and a force of unknown magnitude and direction acting at the pivot, $\vec{F}_p$.

Derivation

We are interested in calculating $\theta(t)$ so we focus on the rotation of the bar about the **pivot** and calculate torque. Since the unknown force $\vec{F}_p$ acts at the pivot, its torque about the pivot is zero (moment-arm has zero length). Therefore only the weight mg appears in our calculations. The torque is given by

$$\tau = -mgl\sin(\theta) \tag{1}$$

where the negative sign denotes the fact that the rotational direction of the torque is always opposite to that of the angle. For instance in the free-body diagram the angle θ is counter-clockwise while the torque exerted by the weight is in the clockwise direction.

The effect of this torque is to produce angular acceleration according the Newton's Second Law of Motion:

$$\tau = I\alpha \tag{2}$$

where I is the moment of inertia of the bar about the pivot and $\alpha = \frac{d^2\theta}{dt^2}$ is its angular acceleration. Note that since I is calculated about the pivot it is a function of the distance l.

Substituting equations (1) and $\alpha = \frac{d^2\theta}{dt^2}$ into (2):

$$\begin{aligned}
\tau &= I\alpha \\
\Rightarrow\ -mgl\sin(\theta) &= I\alpha \\
\Rightarrow\ \alpha &= -\frac{mgl}{I}\sin(\theta) \\
\Rightarrow\ \frac{d^2\theta}{dt^2} &= -\frac{mgl}{I}\sin(\theta)
\end{aligned}$$

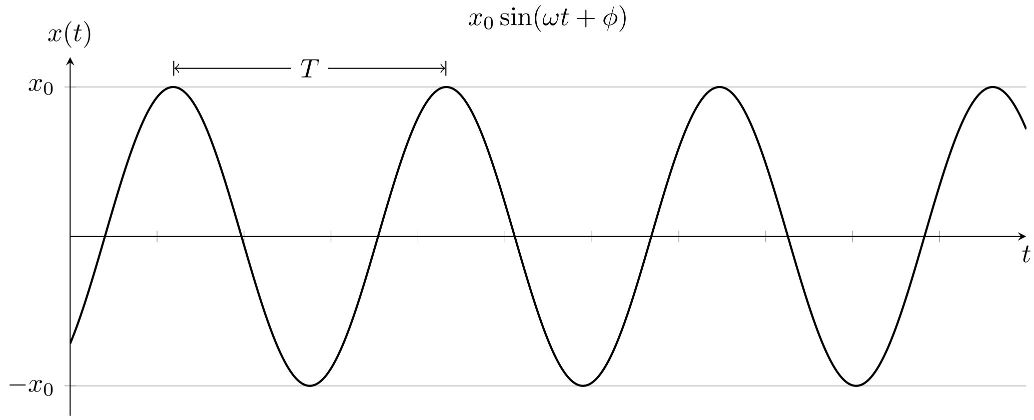
This is a differential equation which needs to be solved for the function $\theta(t)$. In its current form it has no analytical solution. However if we limit the system to small angles $\theta \ll 1$, that is only allow small-angle oscillations (small amplitude) we can make the approximation

$$\theta \ll 1\ \Rightarrow\ \sin(\theta) \approx \theta$$

which transforms the differential equation in to

$$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta \tag{3}$$

This differential equation has the well-known form $\frac{d^2x}{dt^2} = -\omega^2 x$ with the equally well-known solution $x(t) = x_0 \sin(\omega t + \phi)$. $\omega = \frac{2\pi}{T}$ is the angular frequency of the system while T is its time period. The solution $x(t)$ of the differential equation is sketched below. Note its periodic behavior. The motion of the swinging compound pendulum is similar, it swings back and forth, taking the same amount of time (T) to complete each oscillation.



By simply inspecting (3) and comparing with the general form $\frac{d^2x}{dt^2} = -\omega^2 x$ we can conclude that

$$\begin{aligned} \omega^2 &= \frac{mgl}{I} \\ \Rightarrow \ \left(\frac{2\pi}{T}\right)^2 &= \frac{mgl}{I} \\ \Rightarrow \ T &= 2\pi\sqrt{\frac{I}{mgl}} \end{aligned} \tag{4}$$

The final step is to calculate I for a given value of l. For this we use the parallel axis theorem. If the mass of the bar is m then I is given by

$$I = I_{CM} + m\,l^2 \tag{5}$$

where I_{CM} is the moment of inertia of the bar about its center of mass and l is the distance from the pivot to the center of mass. In turn, I_{CM} is calculated by considering the bar to have negligible width and uniform mass distribution, in which case the moment of inertia is known to be given by:

$$I_{CM} = \frac{1}{12}\,m\,L^2 \tag{6}$$

where L is the total length of the bar.

Substituting (5) and (6) in to (4)

$$\begin{aligned} T &= 2\pi\sqrt{\frac{I}{mgl}} \\ \Rightarrow \ T &= 2\pi\sqrt{\frac{I_{CM} + m\,l^2}{mgl}} \\ \Rightarrow \ T &= 2\pi\sqrt{\frac{\frac{1}{12}\,m\,L^2 + m\,l^2}{mgl}} \\ \Rightarrow \ T &= 2\pi\sqrt{\frac{L^2 + 12l^2}{12gl}} \end{aligned} \tag{7}$$

Thus we achieve our objective by deriving an equation that relates the time period T of a compound pendulum to its physical characteristics, mainly its total length L and the distance l from the pivot to the bar's center of mass, and to the acceleration of gravity.

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Objective

Calculate the value of g (acceleration of gravity) and L (the length of the compound pendulum).

Apparatus

- Slotted metal bar
- Suspension bracket
- Stop-watch
- Meter Rule
- Telescope

Procedure

A compound pendulum is a rigid body whose mass is not concentrated at one point and which is capable of oscillating about some fixed pivot (axis of rotation). In this experiment we will be studying the behavior of a uniform metallic bar acting as a compound pendulum. The time-period of the oscillations of a uniform bar is governed by the equation

$$T = 2\pi\sqrt{\frac{L^2 + 12\,l^2}{12\,gl}} \tag{1}$$

where

- T is the time period
- L is the total length of the bar
- g is the acceleration of gravity
- l is the distance from the center of mass of the bar to the pivot

Our aim is to vary l by changing the location of the pivot, and for each value of l measure the time period T. These observations will be used to calculate the acceleration of gravity and the total length of the bar.

Setup

You will be provided a metallic bar with a number of holes/slots placed along its length. Its two ends will be labeled **A** and **B**. The center of mass of the bar will be indicated by a line drawn across its middle. The bar is to be suspended from the wall-mounted bracket using a set of pin and nuts.

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Choose the end of the bar labeled **A**. Pass the pin through the hole/slot closest to this end (furthest away from the center) and use the provided nuts to tighten it in place. Ensure that roughly the same amount of pin protrudes from both ends.

Now suspend the bar from the wall-bracket using the pin. The pin will support the bar and will allow it to oscillate parallel to the wall in the vertical plain.

Place the telescope on a stool and position it so that you can view the suspended bar through it. Adjust the eye-piece (by sliding it) to bring the bar in to focus. Rotate the telescope in place until the cross-hairs are diagonal (no longer aligned with the horizontal and vertical directions). We will use the telescope to count the oscillations of the bar.

Tabulation

Record your data in a table with the following format.

l (cm)	t_{10_1} (s)	t_{10_2} (s)	t_{10_3} (s)	t_{10_4} (s)	t_{10_5} (s)

For each value of l we will measure the time for 10 oscillations, five times, giving us the five t_{10_i}. This table consists of the measurements we take in the experiment. From these measured values we calculate other derived quantities which will allow us to achieve our objective of calculating the values of g and L.

It is recommended that you observe and right down all of the measurements first before you calculate the rest of the values. Using a pencil to write down the values will make it easy to fix inevitable mistakes.

Observation

Start with the pin placed in the top-most hole/slot (next to the end labeled **A**). Use the meter-stick to measure the distance from the Center of Mass of the bar to the center of the pin from which the bar is suspended. This is l. Note down this value in the table.

Start the oscillations by pulling the bar a few degrees (less than 20) out of its stationary vertical position and letting go. Use the telescope to observe the bar swinging past its initial vertical position. This will allow you to count complete oscillations of the pendulum (an oscillation is completed every time the pendulum swings past the initial position moving in the same direction). Use the stop-watch to measure the time taken to complete 10 oscillations. This is t_{10_1}.

Stop the pendulum and then start it swinging again. Take four more measurements of the time taken to complete 10 oscillations. These are t_{10_2}, t_{10_3}, t_{10_4} and t_{10_5}. Note these values in the first table.

Move the pin to the next hole/slot, below the current one. Measure the new value of l and repeat the above procedure to get the five values for the time taken by 10 oscillations.

Keep moving the pin to the next hole/slot until you reach the center of the bar. You will now have your complete set of measurements.

Finally measure L_{actual} the total length of the bar pendulum using the meter-stick.

Calculations

Use your measurements to calculate and record the derived quantities in a table with the following format.

l (cm)	$\bar{t}_{10}$ (s)	$\Delta(t_{10})$ (s)	T (s)	ΔT (s)	l^2 (cm^2)	$T^2 l$ (cm s^2)	$\Delta(T^2 l)$ (cm s^2)

where $\bar{t}_{10}$ is the average/mean of the five t_{10_i} and $\Delta(t_{10})$ is the standard deviation given by

$$\Delta(t_{10}) = \sqrt{\frac{1}{5}\sum_{i=1}^{5}(t_{10_i} - \bar{t}_{10})^2} \tag{2}$$

T (the time for one oscillation) is calculated by dividing the mean time for 10 oscillations (t_{10}) by 10, that is

$$T = \frac{\bar{t}_{10}}{10} \tag{3}$$

The corresponding uncertainty ΔT is given by

$$\Delta T = \frac{\Delta(\bar{t}_{10})}{10} \tag{4}$$

The uncertainty in the derived quantity $T^2\, l$, denoted by $\Delta(T^2\, l)$, comes from this uncertainty ΔT in T. It is calculated using

$$\frac{\Delta(T^2\, l)}{T^2\, l} = 2\frac{\Delta T}{T} \tag{5}$$

where the factor of 2 comes from T being raised to the power 2 in the expression $T^2\, l$.

Graph

The time period of the oscillations of a rigid bar is given by equation (1). This equation is not linear in the dependence of T on l. We transform the equation to get a linear relationship. We start with the original equation.

$$T = 2\pi\sqrt{\frac{L^2 + 12\, l^2}{12\, gl}}$$

We square both sides to remove the square-root on the RHS.

$$T^2 = 4\pi^2\left(\frac{L^2 + 12\, l^2}{12\, gl}\right)$$

$$\Rightarrow\ T^2 = \frac{\pi^2}{3g}\left(\frac{L^2 + 12\, l^2}{l}\right)$$

The equations is still not linear in T^2 and l because of the l in the denominator. We multiply both sides by l.

$$T^2\, l = \frac{\pi^2}{3g}\left(12\, l^2 + L^2\right) \tag{6}$$

If we now consider this equation to be a relationship between $T^2\, l$ and l^2 it is linear. Compare the equation to that of the straight line $y = m\,x + c$ and you can immediately deduce that

$$\begin{aligned} \text{slope} &= \frac{12\pi^2}{3g} = \frac{4\pi^2}{g} \\ y\text{-intercept} &= \frac{\pi^2 L^2}{3g} \end{aligned} \tag{7}$$

Draw a linear graph of $T^2\, l$ vs. l^2 including error bars. *Take special care with your choice of scale. Label all axes clearly.*

Use your graph to calculate the slope and y-intercept of the best-fit line. *Don't forget to write down the units.*

Draw additional steep and shallow fit lines and calculate and note the uncertainty in the slope and y-intercept.

Results

1. Use the calculated value of the slope, its uncertainty and equation (7) to calculate the value of the acceleration of gravity g and its associated uncertainty.

$$\frac{\Delta g}{g} = \frac{\Delta(\text{slope})}{\text{slope}}$$

2. The actual value of g is $981\, cm/s^2$. Compare your calculated value with this by calculating the percentage uncertainty. It is defined as

$$\text{percentage uncertainty} = \frac{|\,\text{actual value} - \text{measured value}\,|}{\text{actual value}} \times 100\% \tag{8}$$

3. Use the calculated value of the y-intercept, its uncertainty, equation (7) and the value of g (calculated from the slope) to calculate the value of L (the length of the bar) and its associated uncertainty.

$$\frac{\Delta L}{L} = \frac{1}{2}\frac{\Delta g}{g} + \frac{1}{2}\frac{\Delta(\text{y-intercept})}{\text{y-intercept}}$$

4. Compare this with the actual value (L_{actual}) measured directly using the meter-stick, by calculating the percentage uncertainty.
5. List the possible sources of error in this experiment.